The Rayleigh-Taylor condition for the evolution of irrotational fluid interfaces.

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Abstract

For the two-phase Hele-Shaw and Muskat problems, and also for irrotational incompressible Euler equation in vacuum, we prove existence locally in time when the Rayleigh-Taylor condition is initially satisfied for a 2D interface. The result for water waves was first obtained by Wu [27] in a slightly different scenario (vanishing at infinity), but our approach is different: it emphasizes the active scalar character of the system and does not require the presence of gravity.

1 Introduction

There are several interesting problems in Fluid Mechanics regarding the evolution of the interface between two fluids (the Hele-Shaw cell [17, 22] and the Muskat problem [20]) or between a fluid and vacuum or another fluid with zero density, such as it happens in the modelization of water waves. In all of them the first important question to be asked is theirs well-possedness, usually within the chain of Sobolev spaces. However such a result turns out to be false for general initial data, but first Rayleigh [21], Taylor [26] and Saffman-Taylor [22], and later Beale-Hou-Lowengrub [4], Wu [27, 28], Christodoulou-Lindblad [7], Ambrose [1], Lindblad [19], Ambrose-Masmoudi [2], Coutand-Shkoller [12], Córdoba-Gancedo [11], Shatah-Zeng [23] and Zhang-Zhang [29] figure out a condition that must be satisfied in order to have a solution locally in time: the different of gradient of pressure of both fluids must be projected with an appropriated sign at the normal vector to the interface. This is known as the Rayleigh-Taylor condition.

In references [9] and [10] we have obtained well-possedness in the 2D case: for the Hele-Shaw and Muskat problems, our result is new in the more difficult case when the two fluids have different densities and viscosities, but for water waves we give a different proof of the important theorem of Wu [27] where gravity plays a crucial role in the sign of the Rayleigh-Taylor condition. In our proof we consider the two cases, with or without gravity, and with initial data also satisfying Rayleigh-Taylor condition.

In some cases for which the Rayleigh-Taylor condition is not satisfied initially there are several proofs of ill-possedness. We point out the works of Ebin [14, 15], Caflisch-Orellana [5], Siegel-Caflisch-Howison [24] and Córdoba-Gancedo [11].

We regard these models as transport equations for the density, considered as an active scalar, with a divergence free velocity field given by Darcy's law (Hele-Shaw and Muskat) or Bernoulli law (irrotational incompressible Euler equation). It follows that the vorticity is given as a delta distribution in the interface multiplied by an amplitude. The dynamic of the interface follows from the Birkhoff-Rott integral of that amplitude to which we may subtract any component in the tangential direction without modifying the interface evolution (see [18]). We treat the case without surface tension which leads to the equality of the pressure on the free boundary. In both problems it is assumed that the initial interface does not touch itself, and we quantify that property imposing that the arch-chord quotient be initially strictly positive. It is part of the evolution problem to check carefully that such a positiveness prevails for a short time (see [16]), as well as the Rayleigh-Taylor condition, depending conveniently upon the initial data.

2 Equations

The free boundary is given by the discontinuity on the densities and the viscosities (in the case of free boundary for the irrotational incompressible Euler equation the viscosity is zero) of the fluids

$$(\mu,\rho)(x_1,x_2,t)=\begin{cases}(\mu^1,\rho^1), & x\in\Omega^1(t)\\ (\mu^2,\rho^2), & x\in\Omega^2(t)=\mathbb{R}^2-\Omega^1(t),\end{cases} \tag{1}$$

and $\mu^1\neq\mu^2$, and $\rho^1\neq\rho^2$ are constants.

Let the free boundary be parameterized by

$$\partial\Omega^j(t)=\{z(\alpha,t)=(z_1(\alpha,t),z_2(\alpha,t)):\alpha\in\mathbb{R}\}$$

such that

$$(z_1(\alpha+2k\pi,t),z_2(\alpha+2k\pi,t))=(z_1(\alpha,t)+2k\pi,z_2(\alpha,t)), \tag{2}$$

with the initial data $z(\alpha,0)=z_0(\alpha)$. Also we shall study the case of a closed curve:

$$(z_1(\alpha+2k\pi,t),z_2(\alpha+2k\pi,t))=(z_1(\alpha,t),z_2(\alpha,t)). \tag{3}$$

We consider that each fluid is irrotational, i.e. $\omega=\nabla\times u=0$, in the interior of each domain Ω^i ($i=1,2$). Therefore the vorticity ω has its support on the curve $z(\alpha,t)$ and it has the form

$$\omega(x,t)=\varpi(\alpha,t)\delta(x-z(\alpha,t)).$$

Then $z(\alpha,t)$ evolves with a velocity field coming from Biot-Savart law, which can be explicitly computed and it is given by the Birkhoff-Rott integral of the amplitude ϖ along the interface curve:

$$BR(z,\varpi)(\alpha,t)=\frac{1}{4\pi}PV\int\frac{(z(\alpha,t)-z(\beta,t))^{\perp}}{|z(\alpha,t)-z(\beta,t)|^2}\varpi(\beta,t)d\beta, \tag{4}$$

for PV principal value (see [25]).

We have

$$\begin{aligned} v^2(z(\alpha,t),t) &= BR(z,\varpi)(\alpha,t) + \frac{1}{2}\frac{\varpi(\alpha,t)}{|\partial_\alpha z(\alpha,t)|^2}\partial_\alpha z(\alpha,t), \\ v^1(z(\alpha,t),t) &= BR(z,\varpi)(\alpha,t) - \frac{1}{2}\frac{\varpi(\alpha,t)}{|\partial_\alpha z(\alpha,t)|^2}\partial_\alpha z(\alpha,t), \end{aligned} \tag{5}$$

where $v^j(z(\alpha,t),t)$ denotes the limit velocity field obtained approaching the boundary in the normal direction inside Ω^j and $BR(z,\varpi)(\alpha,t)$ is given by (4). It gives us the velocity field at the interface to which we can subtract any term in the tangential direction without modifying the geometric evolution of the curve

$$z_t(\alpha,t) = BR(z,\varpi)(\alpha,t) + c(\alpha,t)\partial_\alpha z(\alpha,t). \tag{6}$$

A wise choice of $c(\alpha,t)$ namely:

$$\begin{aligned} c(\alpha,t) = \frac{\alpha+\pi}{2\pi} &\int_{\mathbb{T}} \frac{\partial_\alpha z(\alpha,t)}{|\partial_\alpha z(\alpha,t)|^2}\cdot\partial_\alpha BR(z,\varpi)(\alpha,t)d\alpha \\ &- \int_{-\pi}^{\alpha} \frac{\partial_\alpha z(\beta,t)}{|\partial_\alpha z(\beta,t)|^2}\cdot\partial_\beta BR(z,\varpi)(\beta,t)d\beta, \end{aligned} \tag{7}$$

allows us to accomplish the fact that the length of the tangent vector to $z(\alpha,t)$ be just a function in the variable t only:

$$A(t) = |\partial_\alpha z(\alpha,t)|^2.$$

Next, in order to close the system we apply Darcy's law or Bernoulli law which leads to an equation that relates the parametrization $z(\alpha,t)$ with the amplitude $\varpi(\alpha,t)$.

2.1 Darcy's law

Darcy's law is the following momentum equation for the velocity v

$$\frac{\mu}{\kappa}v = -\nabla p - (0, \mathrm{g}\,\rho), \tag{8}$$

where p is the pressure, μ is the dynamic viscosity, κ is the permeability of the medium, ρ is the liquid density and g is the acceleration due to gravity. Together with the incompressibility condition $\nabla\cdot v = 0$ implies the identity

$$p^2(z(\alpha,t),t) = p^1(z(\alpha,t),t).$$

Let us introduce the following notation:

$$[\mu v](\alpha,t) = (\mu^2 v^2(z(\alpha,t),t) - \mu^1 v^1(z(\alpha,t),t))\cdot\partial_\alpha z(\alpha,t).$$

Then taking the limit in Darcy's law we obtain

$$\begin{aligned} \frac{[\mu v](\alpha,t)}{\kappa} &= -(\nabla p^2(z(\alpha,t),t) - \nabla p^1(z^1(\alpha,t),t))\cdot\partial_\alpha z(\alpha,t) - \mathrm{g}(\rho^2-\rho^1)\,\partial_\alpha z_2(\alpha,t) \\ &= -\partial_\alpha(p^2(z(\alpha,t),t) - p^1(z(\alpha,t),t)) - \mathrm{g}(\rho^2-\rho^1)\,\partial_\alpha z_2(\alpha,t) \\ &= -\mathrm{g}(\rho^2-\rho^1)\,\partial_\alpha z_2(\alpha,t), \end{aligned}$$

which gives us

$$\frac{\mu^2+\mu^1}{2\kappa}\varpi(\alpha,t)+\frac{\mu^2-\mu^1}{\kappa}BR(z,\varpi)(\alpha,t)\cdot\partial_\alpha z(\alpha,t)=-\mathrm{g}(\rho^2-\rho^1)\partial_\alpha z_2(\alpha,t),$$

so that

$$\varpi(\alpha,t)=-A_\mu 2BR(z,\varpi)(\alpha,t)\cdot\partial_\alpha z(\alpha,t)-2\kappa\mathrm{g}\frac{\rho^2-\rho^1}{\mu^2+\mu^1}\partial_\alpha z_2(\alpha,t). \tag{9}$$

where $A_\mu=\frac{\mu_2-\mu_1}{\mu_2+\mu_1}$.

2.2 Bernoulli law

Next we give the evolution equation for the amplitude of vorticity $\varpi(\alpha,t)$ by means of Bernoulli's law. We consider an irrotational flow satisfying Euler equations

$$\rho(v_t+v\nabla v)=-\nabla p-(0,\mathrm{g}\,\rho), \tag{10}$$

and the incompressibility condition $\nabla\cdot v=0$. Denote ϕ such that $v(x,t)=\nabla\phi(x,t)$, then we have the expression

$$\rho(\phi_t(x,t)+\frac{1}{2}|v(x,t)|^2+\mathrm{g}x_2)+p(x,t)=0.$$

From Biot-Savart law, for $x\neq z(\alpha,t)$, we get

$$\phi(x,t)=\frac{1}{2\pi}PV\int\arctan\Big(\frac{x_2-z_2(\beta,t)}{x_1-z_1(\beta,t)}\Big)\varpi(\beta,t)d\beta.$$

Let us define

$$\Pi(\alpha,t)=\phi^2(z(\alpha,t),t)-\phi^1(z(\alpha,t),t),$$

where again $\phi^j(z(\alpha,t),t)$ denotes the limit obtained approaching the boundary in the normal direction inside Ω^j. It is clear that

$$\begin{aligned}\partial_\alpha\Pi(\alpha,t)&=(\nabla\phi^2(z(\alpha,t),t)-\nabla\phi^1(z(\alpha,t),t))\cdot\partial_\alpha z(\alpha,t)\\&=(v^2(z(\alpha,t),t)-v^1(z(\alpha,t),t))\cdot\partial_\alpha z(\alpha,t)\\&=\varpi(\alpha,t),\end{aligned}$$

therefore

$$\int_{\mathbb{T}}\varpi(\alpha,t)d\alpha=0.$$

Now we observe that

$$\begin{aligned}\phi^2(z(\alpha,t),t)&=IT(z,\varpi)(\alpha,t)+\frac{1}{2}\Pi(\alpha,t),\\ \phi^1(z(\alpha,t),t)&=IT(z,\varpi)(\alpha,t)-\frac{1}{2}\Pi(\alpha,t).\end{aligned} \tag{11}$$

where
$$IT(z,\varpi)(\alpha,t)=\frac{1}{2\pi}PV\int \arctan\Big(\frac{z_2(\alpha,t)-z_2(\beta,t)}{z_1(\alpha,t)-z_1(\beta,t)}\Big)\varpi(\beta,t)d\beta.$$

Then using Bernoulli law inside each domain and taking limits approaching the common boundary, one finds
$$\rho^j(\phi^j_t(z(\alpha,t),t)+\frac{1}{2}|v^j(z(\alpha,t),t)|^2+\mathrm{g}z_2(\alpha,t))+p^j(z(\alpha,t),t)=0,$$
and since
$$p^1(z(\alpha,t),t)=p^2(z(\alpha,t),t),$$
we get
$$[\rho\phi_t](\alpha,t)+\frac{\rho^2}{2}|v^2(z(\alpha,t),t)|^2-\frac{\rho^1}{2}|v^1(z(\alpha,t),t)|^2+(\rho^2-\rho^1)\mathrm{g}z_2(\alpha,t)=0 \tag{12}$$
where we have introduced the following notation:
$$[\rho\phi_t](\alpha,t)=\rho^2\phi^2_t(z(\alpha,t),t)-\rho^1\phi^1_t(z(\alpha,t),t).$$
Then it is clear that $\phi^j_t(z(\alpha,t),t)=\partial_t(\phi^j(z(\alpha,t),t))-z_t(\alpha,t)\cdot\nabla\phi^j(z(\alpha,t),t)$, and using (11) we find
$$\begin{aligned}[\rho\phi_t](\alpha,t)=&\frac{\rho^2+\rho^1}{2}\Pi_t(\alpha,t)+(\rho^2-\rho^1)\partial_t(IT(z,\varpi)(\alpha,t))\\ &-z_t(\alpha,t)\cdot(\rho^2v^2(z(\alpha,t),t)-\rho^1v^1(z(\alpha,t),t)).\end{aligned}$$
The equations (5) and (6) in (12) give
$$\begin{aligned}\Pi_t(\alpha,t)=&-2A_\rho\partial_t(IT(z,\varpi)(\alpha,t))+c(\alpha,t)\varpi(\alpha,t)+A_\rho|BR(z,\varpi)(\alpha,t)|^2\\ &+2A_\rho c(\alpha,t)BR(z,\varpi)(\alpha,t)\cdot\partial_\alpha z(\alpha,t)-A_\rho\frac{|\varpi(\alpha,t)|^2}{4|\partial_\alpha z(\alpha,t)|^2}-2A_\rho\mathrm{g}z_2(\alpha,t).\end{aligned}\tag{13}$$
where $A_\rho=\frac{\rho_2-\rho_1}{\rho_2+\rho_1}$.

Easily we find the identity:
$$\begin{aligned}\partial_\alpha\partial_t(IT(z,\varpi)(\alpha,t))=&\,\partial_t(BR(z,\varpi)(\alpha,t)\cdot\partial_\alpha z(\alpha,t))\\ =&\,\partial_t(BR(z,\varpi)(\alpha,t))\cdot\partial_\alpha z(\alpha,t)+BR(z,\varpi)(\alpha,t)\cdot\partial_\alpha BR(z,\varpi)(\alpha,t)\\ &+c(\alpha,t)BR(z,\varpi)(\alpha,t)\cdot\partial^2_\alpha z(\alpha,t)+\partial_\alpha c(\alpha,t)BR(z,\varpi)(\alpha,t)\cdot\partial_\alpha z(\alpha,t)\end{aligned}$$
Then taking a derivative in (13) and using the identity above we get the desired formula for ϖ, which in the case $A_\rho=1$, i.e. $\rho_1=0$, reads as follows
$$\begin{aligned}\varpi_t(\alpha,t)=&-2\partial_t BR(z,\varpi)(\alpha,t)\cdot\partial_\alpha z(\alpha,t)-\partial_\alpha(\frac{|\varpi|^2}{4|\partial_\alpha z|^2})(\alpha,t)+\partial_\alpha(c\,\varpi)(\alpha,t)\\ &+2c(\alpha,t)\partial_\alpha BR(z,\varpi)(\alpha,t)\cdot\partial_\alpha z(\alpha,t)+2\mathrm{g}\partial_\alpha z_2(\alpha,t).\end{aligned}\tag{14}$$
That is the standard water waves model where g is the acceleration due to gravity.

3 Rayleigh-Taylor condition

Our next step is to find the formula for the difference of the gradients of the pressure in the normal direction:

$$\sigma(\alpha,t) = -(\nabla p^2(z(\alpha,t),t) - \nabla p^1(z(\alpha,t),t)) \cdot \partial_\alpha^\perp z(\alpha,t).$$

3.1 Darcy's law

Approaching the boundary in Darcy's law, we get

$$\sigma(\alpha,t) = \frac{\mu^2-\mu^1}{\kappa} BR(z,\varpi)(\alpha,t) \cdot \partial_\alpha^\perp z(\alpha,t) + \mathrm{g}(\rho^2-\rho^1)\partial_\alpha z_1(\alpha,t).$$

3.2 Bernoulli law

We will consider the case $A_\rho = 1$. This gives in the Euler equation $-\nabla p(x,t) = 0$ inside $\Omega^1(t)$ and therefore $\nabla p^1(z(\alpha,t),t) = 0$. Next we define the Lagrangian coordinates for the free boundary with the velocity v^2

$$\begin{aligned} Z_t(\gamma,t) &= v^2(Z(\gamma,t),t)) \\ Z(\gamma,0) &= z_0(\gamma). \end{aligned}$$

We have the same curve with different parameterizations $Z(\gamma,t) = z(\alpha(\gamma,t),t)$ and two equations for the velocity of the curve, namely

$$\begin{aligned} Z_t(\gamma,t) &= z_t(\alpha,t) + \alpha_t(\gamma,t)\partial_\alpha z(\alpha,t) \\ &= BR(z,\varpi)(\alpha,t) + c(\alpha,t)\partial_\alpha z(\alpha,t) + \alpha_t(\gamma,t)\partial_\alpha z(\alpha,t) \end{aligned} \tag{15}$$

and another given by

$$Z_t(\gamma,t) = BR(z,\varpi)(\alpha,t) + \frac{1}{2}\frac{\varpi(\alpha,t)}{|\partial_\alpha z(\alpha,t)|^2}\partial_\alpha z(\alpha,t). \tag{16}$$

Define the function $\varphi(\alpha,t)$ (see [4] and [2]) by

$$\varphi(\alpha,t) = \frac{1}{2}\frac{\varpi(\alpha,t)}{|\partial_\alpha z(\alpha,t)|} - c(\alpha,t)|\partial_\alpha z(\alpha,t)|. \tag{17}$$

The dot product of equations (16) and (15) with the tangential vector gives

$$\alpha_t(\gamma,t) = \frac{\varphi(\alpha,t)}{|\partial_\alpha z(\alpha,t)|}. \tag{18}$$

Taking a time derivative in (16) yields

$$\begin{aligned} Z_{tt}(\gamma,t)\cdot\partial_\alpha^\perp z(\alpha,t) &= (\partial_t BR(z,\varpi)(\alpha,t) + \alpha_t(\gamma,t)\partial_\alpha BR(z,\varpi)(\alpha,t)) \cdot \partial_\alpha^\perp z(\alpha,t) \\ &\quad + \frac{1}{2}\frac{\varpi(\alpha,t)}{|\partial_\alpha z(\alpha,t)|^2}(\partial_\alpha z_t(\alpha,t) + \alpha_t(\gamma,t)\partial_\alpha^2 z(\alpha,t)) \cdot \partial_\alpha^\perp z(\alpha,t) \end{aligned}$$

and therefore

$$
\begin{aligned}
\sigma(\alpha,t) &= (\partial_t BR(z,\varpi)(\alpha,t) + \frac{\varphi(\alpha,t)}{|\partial_\alpha z(\alpha,t)|}\partial_\alpha BR(z,\varpi)(\alpha,t)) \cdot \partial_\alpha^\perp z(\alpha,t) \\
&\quad + \frac{1}{2}\frac{\varpi(\alpha,t)}{|\partial_\alpha z(\alpha,t)|^2}(\partial_\alpha z_t(\alpha,t) + \frac{\varphi(\alpha,t)}{|\partial_\alpha z(\alpha,t)|}\partial_\alpha^2 z(\alpha,t)) \cdot \partial_\alpha^\perp z(\alpha,t) + \mathrm{g}\partial_\alpha z_1(\alpha,t).
\end{aligned}
\tag{19}
$$

4 Local existence

Our main results consist on the existence of a positive time T (depending upon the initial conditions) for which we have a solution of the periodic Muskat problem (equations (6), (7) and (9)) and of the free boundary of the irrotational incompressible Euler equations in vacuum (equations (6), (7) and (14)) during the time interval $[0,T]$ so long as the initial data belong to $H^k(\mathbb{T})$ for k sufficiently large, $\mathcal{F}(z_0)(\alpha,\beta) < \infty$, and

$$\sigma_0(\alpha) = -(\nabla p^2(z_0(\alpha)) - \nabla p^1(z_0(\alpha))) \cdot \partial_\alpha^\perp z_0(\alpha) > 0,$$

where p^j denote the pressure in Ω^j and the function $\mathcal{F}(z)$, which measures the arc-chord condition (see [16]), is defined by

$$\mathcal{F}(z)(\alpha,\beta,t) = \frac{|\beta|}{|z(\alpha,t) - z(\alpha-\beta,t)|} \qquad \forall\, \alpha,\beta \in (-\pi,\pi), \tag{20}$$

with

$$\mathcal{F}(z)(\alpha,0,t) = \frac{1}{|\partial_\alpha z(\alpha,t)|}.$$

Theorem 4.1 *Let $z_0(\alpha) \in H^k(\mathbb{T})$ for $k \geq 3$, $\mathcal{F}(z_0)(\alpha,\beta) < \infty$, and*

$$\sigma_0(\alpha) = -(\nabla p^2(z_0(\alpha)) - \nabla p^1(z_0(\alpha))) \cdot \partial_\alpha^\perp z_0(\alpha) > 0.$$

Then there exists a time $T > 0$ so that there is a solution to (6), (7) *and* (9) *in $C^1([0,T]; H^k(\mathbb{T}))$ with $z(\alpha,0) = z_0(\alpha)$.*

Theorem 4.2 *Let $z_0(\alpha) \in H^k(\mathbb{T})$, $\varphi(\alpha,0) = \varphi_0(\alpha) \in H^{k-\frac{1}{2}}$ defined in* (17) *for $k \geq 4$, $\mathcal{F}(z_0)(\alpha,\beta) < \infty$, $\mathrm{g} \geq 0$, and*

$$\sigma_0(\alpha) = -(\nabla p^2(z_0(\alpha)) - \nabla p^1(z_0(\alpha))) \cdot \partial_\alpha^\perp z_0(\alpha) > 0.$$

Then there exists a time $T > 0$ so that there is a solution to (6), (7) *and* (14) *with $z(\alpha,t) \in C^1([0,T]; H^k(\mathbb{T}))$, $\varpi(\alpha,t) \in C^1([0,T]; H^{k-1}(\mathbb{T}))$ for $z(\alpha,0) = z_0(\alpha)$ and $\varpi(\alpha,0) = \varpi_0(\alpha)$.*

Remark 4.3 *Notice that the parametrization is defined with properties* (2) *or* (3). *But in the Hele-Shaw and Muskat problems we can show easily that*

$$\int_{-\pi}^{\pi} \sigma(\alpha,t)d\alpha = 0$$

for a closed curve, making impossible the task of prescribing a sign to the Rayleigh-Taylor condition.

5 Sketch of the proof

First we consider the operator $T(u)(\alpha) = 2BR(z,u)(\alpha) \cdot \partial_\alpha z(\alpha)$ associated to a smooth H^3 curve z satisfying the arch-chord condition. T is a smoothing compact operator in Sobolev space whose adjoint T^*, acting on u, can be described in term of the Cauchy integral of u along the curve z and it is shown that their eigenvalues have absolute value strictly less than one (see [3]).

In our proof it is crucial to get control of the norm of the inverse operators $(I - \xi T)^{-1}$, $|\xi| \leq 1$. The arguments rely upon the boundedness properties of the Hilbert transforms associated to $C^{1,\alpha}$ curves, for which we need precise estimates obtained with arguments involving conformal mappings, Hopf maximum principle and Harnack inequalities (see [6] and [13]).

We then provide upper bounds for the amplitude of the vorticity, the Birkhoff-Rott integral, the parametrization of the curve, the arc-chord condition and the Rayleigh-Taylor condition, namely:

5.1 A priori estimates for theorem 4.1

$$\|\varpi\|_{H^k} \leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty} + \|z\|^2_{H^{k+1}}),$$

$$\|BR(z,\varpi)\|_{H^k} \leq \exp(C(\|\mathcal{F}(z)\|^2_{L^\infty} + \|z\|^2_{H^{k+1}}),$$

$$\begin{aligned}\frac{d}{dt}\|z\|^2_{H^k}(t) \leq -\frac{\kappa}{2\pi(\mu_1+\mu_2)} \int_{\mathbb{T}} \frac{\sigma(\alpha,t)}{|\partial_\alpha z(\alpha)|^2}\partial^k_\alpha z(\alpha,t)\cdot\Lambda(\partial^k_\alpha z)(\alpha,t)d\alpha \\ + \exp C(\|\mathcal{F}(z)\|^2_{L^\infty}(t) + \|z\|^2_{H^k}),\end{aligned}$$

and

$$\frac{d}{dt}\|\mathcal{F}(z)\|^2_{L^\infty}(t) \leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty}(t) + \|z\|^2_{H^3}(t))$$

where the operator Λ is defined by the Fourier transform $\widehat{\Lambda f}(\xi) = |\xi|\widehat{f}(\xi)$ and $\sigma(\alpha,t)$ is the difference of the gradients of the pressure in the normal direction. Finally we study the evolution of $m(t) = \min_{\alpha\in\mathbb{T}} \sigma(\alpha,t)$ (see [8]), which satisfies the following lower bound

$$m(t) \geq m(0) - \int_0^t \exp C(\|\mathcal{F}(z)\|^2_{L^\infty}(s) + \|z\|^2_{H^3}(s))ds.$$

5.2 A priori estimates for theorem 4.2

$$\|BR(z,\varpi)\|_{H^k} \leq C(\|\mathcal{F}(z)\|^2_{L^\infty} + \|z\|^2_{H^{k+1}} + \|\varpi\|^2_{H^k})^m,$$

$$\|z_t\|_{H^k} \leq C(\|\mathcal{F}(z)\|^2_{L^\infty} + \|z\|^2_{H^{k+1}} + \|\varpi\|^2_{H^k})^m,$$

$$\|\varpi_t\|_{H^k} \leq C\big(\exp C(\|\mathcal{F}(z)\|^2_{L^\infty} + \|z\|^2_{H^3})\big)(\|\mathcal{F}(z)\|^2_{L^\infty} + \|z\|^2_{H^{k+2}} + \|\varpi\|^2_{H^{k+1}} + \|\varphi\|^2_{H^{k+1}})^m,$$

$$\|\varpi\|_{H^k} \leq C(\|\mathcal{F}(z)\|^2_{L^\infty} + \|z\|^2_{H^{k+1}} + \|\varpi\|^2_{H^{k-1}} + \|\varphi\|^2_{H^k})^m.$$

We define

$$E(t) = \|z\|^2_{H^{k-1}}(t) + \int_{\mathbb{T}} \frac{2\sigma(\alpha,t)}{|\partial_\alpha z(\alpha,t)|^2}|\partial^k_\alpha z(\alpha,t)|^2 d\alpha + \|\mathcal{F}(z)\|^2_{L^\infty}(t) + \|\varpi\|^2_{H^{k-2}}(t) + \|\varphi\|^2_{H^{k-\frac{1}{2}}}(t)$$

and as before we take $m(t) = \min_{\alpha\in\mathbb{T}} \sigma(\alpha,t)$. Therefore using that

$$\|\partial^4_\alpha z\|^2_{L^2}(t) = \int_{\mathbb{T}} \frac{\sigma(\alpha,t)}{\sigma(\alpha,t)}|\partial^4_\alpha z(\alpha,t)|^2 d\alpha \leq \frac{1}{m(t)}\int_{\mathbb{T}} \sigma(\alpha,t)|\partial^4_\alpha z(\alpha,t)|^2 d\alpha,$$

we obtain

$$\frac{d}{dt}E(t) \leq \frac{1}{m^p(t)} C\exp(CE(t)),$$

with $p \in \mathbb{N}$. It yields

$$E(t) \leq -\frac{1}{C}\ln\big(\exp(-CE(0)) - C^2\int_0^t \frac{1}{m^p(s)}ds\big). \tag{21}$$

The same argument (see [8]) allows us to accomplish the fact that $m'(t) = \sigma_t(\alpha_t, t)$ for almost all t, and formula (19) gives

$$\|\sigma_t(\alpha_t,t)\|_{L^\infty} \leq \frac{1}{m^p(t)} C\exp(CE(t)),$$

and one can find

$$m(t) \geq m(0) + \frac{1}{C}\ln\big(1 - \frac{C^2}{\exp(-CE(0))}\int_0^t \frac{1}{m^p(s)}ds\big).$$

5.3 Existence

To conclude the existence proof we introduce regularized evolution equations satisfying uniformly the a priori estimates above (allowing us to take limits) and for which the local existence follows by standard arguments. Furthermore, in the case of the Hele-Shaw and Muskat problem, in order to take advantage of the positivity of σ we use the pointwise inequality satisfied by the non-local operator Λ (see [8]).

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